THE PROUHET-TARRY-ESCOTT PROBLEM
AND
GENERALIZED THUE-MORSE SEQUENCES

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ABSTRACT. We present new methods of generating Prouhet-Tarry-Escott partitions of arbitrarily large regularity. One of these methods generalizes the construction of the Thue-Morse sequence to finite alphabets with more than two letters. We show how one can use such partitions to (theoretically!) pour the same volume coffee from an urn into a finite number of cups so that each cup gets almost the same amount of caffeine.

CONTENTS

1. INTRODUCTION

Mathematicians have studied the eponymous objects in our title for more than a century and a half. We've stumbled on some generalizations with interesting consequences and new open questions.

Our contribution to the ongoing story began with a query from Richman asking about how he might generalize his solution [Ric01] to the problem of pouring two cups of coffee of equal strength from a carafe in which the concentration increases with depth to three or more cups.

To fill two cups with four pours use the *word* ABBA: pour the first and last quarters into cup A and the second and third quarters into B.

For eight pours the magic word is `ABBA BAAB`. Continuing recursively by appending to each sequence of length n its complement (in the obvious sense) you find the optimal partitions for pourings using 2^k subdivisions. Collecting all the solutions into the infinite word

$$\texttt{AB BA BAAB BAABABBA} \ \ldots$$

produces the *Thue-Morse sequence*.

Richman's argument showing (for example) that the word `ABBA BAAB` solves the two cup problem using eighths depends essentially on

$$1 + 4 + 6 + 7 = 2 + 3 + 5 + 8$$

and

$$1^2 + 4^2 + 6^2 + 7^2 = 2^2 + 3^2 + 5^2 + 8^2.$$

These equations say that the partition $\{\{1, 4, 6, 7\}, \{2, 3, 5, 8\}\}$ whose blocks are the positions of `A` and `B` in the magic word solves an instance of the *Prouhet-Tarry-Escott problem* - finding partitions of a set of integers such that each block has the same sum of powers for several powers.

It's this connection we will generalize.

2. Words and partitions

We set the stage with some formal definitions.

Definition 2.1. *Let S be a non empty set of integers and $r \geqslant -1$ an integer. A partition $P = \{S_1, \ldots, S_b\}$ of S is r-regular if*

$$\sum_{x \in S_1} x^k = \sum_{x \in S_2} x^k = \cdots = \sum_{x \in S_b} x^k$$

for all $k = 0, 1, \ldots, r$. We write $\mathrm{PTE}(S, b, r)$ for the set of all such partitions. A partition P has maximal regularity r *if it is r-regular but not $(r+1)$-regular.*

Remark 2.2. *Every partition is (-1)-regular, so $\mathrm{PTE}(S, b, -1)$ is the set of partitions of S into b blocks. Some of the blocks may be empty.*

Remark 2.3. *This definition and much of what follows makes sense over any ring, not just $\mathbb{Z}$.*

If $P = \{S_1, \ldots, S_b\}$ is an r-regular partition of S with $r \geqslant 0$ then its blocks have the same number of elements, and therefore b divides $m = \#S$. Clearly

$$\emptyset = \mathrm{PTE}(S, b, m/b) \subseteq \cdots \subseteq \mathrm{PTE}(S, b, 1) \subseteq \mathrm{PTE}(S, b, 0).$$

Lemma 2.4. *(Affine invariance) Let $n \neq 0$ and a be integers. Define $f\colon \mathbb{Z} \to \mathbb{Z}$ by $f(x) = a + nx$. If $P = \{S_1, \ldots, S_b\}$ partitions S then*

$$a + nP := f(P) = \{f(S_1), \ldots, f(S_b)\}$$

partitions $a + nS$, and

$$P \in \mathrm{PTE}(S, b, r) \iff a + nP \in \mathrm{PTE}(a + nS, b, r) \ .$$

Proof. An easy induction on the powers less than or equal to r. $\square$

In other words, regularity is invariant under affine transformations.

We are interested in the Prouhet-Tarry-Escott problem when S is a set of consecutive integers. Affine invariance implies that we need consider just $S = [m] = \{1, \ldots, m\}$; we will write $\mathrm{PTE}(m, b, r)$ for $\mathrm{PTE}([m], b, r)$. In that case, b-block partitions have natural string representations over an alphabet $\mathcal{A}$ with b letters $a_1, \ldots, a_b$.

Definition 2.5. *The* string representation *of a b-block partition $P = (S_1, \ldots, S_b)$ of $S = [m]$ is the m-letter word $a_1 a_2 \ldots a_m$ where a_i is the tth letter of the alphabet when $i \in S_t$.*

Conversely, given an m-letter word w on a b letter alphabet we can construct the partition P_w of $[m]$ using the equivalence relation that defines two indices as equivalent when w has the same letter in those two places.

For the letters in reasonably small alphabets we will use $\mathtt{A}, \mathtt{B}, \mathtt{C}, \ldots$ rather than subscripts a_i or integers. We may also occasionally leave blanks between the letters to emphasize features of interest. These have no semantic significance.

In what follows we will freely interchange partitions of $[m]$ and the corresponding words. Some arguments are better in one language, some in the other.

Permuting the letters of the alphabet corresponds to permuting the order in which we write the blocks of the partition. Since that order is essentially irrelevant, we will usually impose a particular lexicographic order on the alphabet, and use letters in that order as necessary starting at the beginning of a word.

In some studies of the Thue-Morse sequence and its generalizations it's convenient to use the alphabet $\{0, 1, \ldots, m-1\}$. If you number the blocks of the partition with those digits rather than those in $[m]$ then the m-letter words that encode the partitions can be viewed as integers written in base m.

3. A NEW CLASS OF SOLUTIONS

In this section we generalize the recursive construction of the Thue-Morse sequence in order to generate a new family of solutions to our Prouhet-Tarry-Escott problems.

Definition 3.1. *A Latin square on a b-letter alphabet is a $b \times b$ square matrix of letters such that each letter occurs exactly once in each row and each column. When we fix an order on the alphabet, a Latin square is* normalized *when its first column is in alphabetical order.*

A Latin square can always be normalized by permuting its rows. In the literature "normalized" sometimes means the columns are permuted as well so that the first row is in alphabetical order. We do not require that.

Example 3.2. *There is only one normalized Latin square on a 2-letter alphabet,*

$$L_0 = \begin{bmatrix} A & B \\ B & A \end{bmatrix} .$$

There are two normalized Latin squares on a 3-letter alphabet:

$$L_1 = \begin{bmatrix} A & B & C \\ B & C & A \\ C & A & B \end{bmatrix} \quad and \quad L_2 = \begin{bmatrix} A & C & B \\ B & A & C \\ C & B & A \end{bmatrix} .$$

The columns of a normalized Latin square L of size b correspond to a sequence of permutations $(\pi_1 = \mathrm{id}, \pi_2, \ldots, \pi_b)$ such that for each row x of L the sequence $(\pi_1(x) = x, \pi_2(x), \ldots, \pi_b(x))$ is a permutation of the alphabet. We will often use that list of permutations to represent L:

$$L = (\mathrm{id}, \pi_2, \ldots, \pi_b).$$

Now we use normalized Latin squares to capture the essence of the recursive construction of the Thue-Morse sequence.

Definition 3.3. *If $w = a_1 a_2 \ldots a_m$ is an $m-$letter word and π is a permutation of the alphabet, then $\pi(w)$ is the m-letter word*

$$\pi(w) = \pi(a_1)\pi(a_2) \ldots \pi(a_m) .$$

When $L = (id, \pi_2, \ldots, \pi_b)$ is a normalized Latin square we write $L(w)$ for the concatenated mb-letter word

$$L(w) = w\pi_2(w) \cdots \pi_b(w) .$$

If P is the partition corresponding to word w then we write $L(P)$ for the partition corresponding the word $L(w)$.

Example 3.4. *With the notations of Example 3.2,*

$$(3.1) \qquad L_0(\texttt{ABBA BAAB}) = \texttt{ABBABAAB BAABABBA}$$

and

$$(3.2) \qquad L_2(\texttt{AB}) = \texttt{ABCABC}.$$

The motivation for Definition 3.3 is the fact that using a Latin square this way increases the regularity of a partition. See [AS99], [Leh47] for references to Prouhet's construction, based on a Latin square action on a cycle of maximal length.

The partition corresponding to the word $\texttt{AB}$ on the alphabet $\{\texttt{A}, \texttt{B}, \texttt{C}\}$ is just (-1)-regular; Equation (3.2) shows that it extends to word $\texttt{ABCABC}$, which corresponds to a 0-regular partition.

The example in Equation (3.1) is more interesting. The word on the left encodes a 2-regular partition. The one on the right corresponds to

$$\{\{1, 4, 6, 7, 10, 11, 13, 16\}, \{2, 3, 5, 8, 9, 12, 14, 15\}\},$$

which is 3-regular. Here's the last step in the proof, assuming we've already showed that it's 2-regular. Let X be the sum of the cubes in the first block:

$$
\begin{aligned}
X &= 1^3 + 4^3 + 6^3 + 7^3 + 10^3 + 11^3 + 13^3 + 16^3 \\
&= 1^3 + 4^3 + 6^3 + 7^3 + (2+8)^3 + (3+8)^3 + (5+8)^3 + (8+8)^3 \\
&= 1^3 + 4^3 + 6^3 + 7^3 \\
&\quad + 2^3 + 3(2^2 \times 8) + 3(2 \times 8^2) + 8^3 \\
&\quad + 3^3 + 3(3^2 \times 8) + 3(3 \times 8^2) + 8^3 \\
&\quad + 5^3 + 3(5^2 \times 8) + 3(5 \times 8^2) + 8^3 \\
&\quad + 8^3 + 3(8^2 \times 8) + 3(8 \times 8^2) + 8^3 \\
&= \sum_{k=1}^{8} k^3 + 24(2^2 + 3^2 + 5^2 + 8^2) + 192(2 + 3 + 5 + 8) + 4(8^3).
\end{aligned}
$$

The same kind of computation shows that the sum Y of the cubes in the second block is

$$Y = \sum_{k=1}^{8} k^3 + 24(1^2 + 4^2 + 6^2 + 7^2) + 192(1 + 4 + 6 + 7) + 4(8^3).$$

Since the partition corresponding to $\texttt{ABBABAAB}$ is 2-regular, $X = Y$.

The formal proof of the general theorem calls for some machinery that's a little more intricate than we like.

Definition 3.5. *Let L be a Latin square. Define its* encoding matrix *$M = \mathcal{E}(L)$ by*

$$M_{ij} = x \iff L_{jx} = i.$$

Thus M_{ij} is the index of the column of L in which the entry i occurs on row j:

$$L_{j,M_{i,j}} = i \iff M_{L_{i,j},i} = j.$$

Example 3.6. *If*

$$L = \begin{bmatrix} A & B & C \\ B & C & A \\ C & A & B \end{bmatrix}$$

then

$$M = \mathcal{E}(L) = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

Theorem 3.7. *Suppose P partitions $[m]$ into b blocks and L is a normalized Latin square of size b.*

(1) *If P is r-regular, then $L(P)$ is $(r+1)$-regular.*
(2) *If the encoding matrix $M = \mathcal{E}(L)$ is invertible and $L(P)$ is $(r+1)$-regular then P is r-regular.*
(3) *If $\mathcal{E}(L)$ is not invertible, then there exist partitions P such that $L(P)$ is 1-regular but P is not 0-regular.*

Proof. Let $w = w_1 w_2 \ldots w_m$ be the word corresponding to P on the alphabet $\mathcal{A} = \{a_1, a_2, \ldots, a_b\}$. For $j \geqslant 0$ and $x \in \mathcal{A}$ let

$$S_{w,x}^{(j)} = \sum \{t^j \mid w_t = x, 1 \leqslant t \leqslant m\}.$$

Then P is r-regular if and only if for every $j = 0, \ldots, r$, the sum $S_{w,x}^{(j)}$ is the same for all $x \in \mathcal{A}$.

Then

$$S^{(j)}_{L(w),x} = \sum \left\{ t^j \mid L(w)_t = x, 1 \leqslant t \leqslant bm \right\}$$

$$= \sum_{k=0}^{b-1} \left[\sum \left\{ (km+t)^j \mid L(w)_{km+t} = x, 1 \leqslant t \leqslant m \right\} \right]$$

$$= \sum_{k=0}^{b-1} \left[\sum \left\{ (km+t)^j \mid \pi_{k+1}(w_t) = x, 1 \leqslant t \leqslant m \right\} \right]$$

$$= \sum_{k=0}^{b-1} \left[\sum \left\{ \sum_{i=0}^{j} \binom{j}{i} (km)^{j-i} t^i \mid w_t = \pi^{-1}_{k+1}(x), 1 \leqslant t \leqslant m \right\} \right]$$

$$= \sum_{i=0}^{j} \sum_{k=0}^{b-1} \binom{j}{i} (km)^{j-i} S^{(i)}_{w,\pi^{-1}_{k+1}(x)} \ .$$

Setting $x = a_s \in \mathcal{A}$,

$$\pi^{-1}_{k+1}(a_s) = a_q \iff s = \pi_{k+1}(q) \iff L_{q,k+1} = s \iff k+1 = M_{sq}.$$

Hence

$$S^{(j)}_{L(w),a_s} = \sum_{q=1}^{b} S^{(j)}_{w,a_q} + jm \sum_{q=1}^{b} (M_{sq} - 1) S^{(j-1)}_{w,a_q}$$

$$+ \sum_{i=0}^{j-2} \sum_{k=0}^{b-1} \binom{j}{i} (km)^{j-i} S^{(i)}_{w,\pi^{-1}_{k+1}(a_s)}$$

(3.3)

$$= X(m,j) + jm \sum_{q=1}^{b} M_{sq} S^{(j-1)}_{w,a_q}$$

$$+ \sum_{i=0}^{j-2} \sum_{k=0}^{b-1} \binom{j}{i} (km)^{j-i} S^{(i)}_{w,\pi^{-1}_{k+1}(a_s)} \ ,$$

where

$$X(m,j) = \sum_{k=1}^{m} (k^j - jmk^{j-1})$$

is independent of w and s.

If w is r-regular then for every $i = 0, \ldots, r$, the sum $S^{(i)}_{w,y}$ is independent of y. Then for all $j = 0, \ldots, r+1$, the sum $S^{(j)}_{L(w),a_s}$ does not depend on a_s, which means that $L(w)$ has regularity $r+1$.

To prove (2), suppose that $L(w)$ is $(r+1)$-regular and M is invertible. There's nothing to prove if $r = 0$, so we start with $r = 1$.

Let $Y_w^{(j)}$ be the column vector with entries $S_{w,x}^{(j)}$ for $x \in \mathcal{A}$ and E the column vector with b entries, all equal to 1. Then (3.3) implies

$$Y_{L(w)}^{(1)} - X(m,1)E = mMY_w^{(0)} .$$

If $L(w)$ is 1-regular, then the left hand side is a multiple of E. Since E is an eigenvector of M, if M is invertible, then the right hand side must also be a multiple of E, which shows that w is 0-regular. Induction on r using the same argument completes the proof of the second statement.

For (3), suppose that M is not invertible. Then its columns are linearly dependent, so we can find integers $c_1, \ldots, c_b$ such that

$$c_1 \mathrm{Col}_1(M) + \cdots + c_b \mathrm{Col}_b(M) = 0 .$$

Since the entries of M are strictly positive, there will be both strictly positive and strictly negative values among $c_1, \ldots, c_b$. Pick a positive integer h such that all the values $h + c_1, \ldots, h + c_b$ are non-negative and consider any word w with $h + c_1$ letters a_1, $h + c_2$ letters a_2, and so on. Then w is not 0-regular, but $L(w)$ is 1-regular. $\qquad\square$

For example, let L be the Latin square

$$\text{(3.4)} \qquad L = \begin{bmatrix} \text{A} & \text{B} & \text{C} & \text{D} \\ \text{B} & \text{A} & \text{D} & \text{C} \\ \text{C} & \text{D} & \text{A} & \text{B} \\ \text{D} & \text{C} & \text{B} & \text{A} \end{bmatrix} \simeq \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} ,$$

corresponding to the multiplication table for the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In this example the matrix $\mathcal{E}(L)$ is the same as L and is not invertible. An example of a linear relation among the columns of $\mathcal{E}(L)$ is

$$\mathrm{Col}_1 - \mathrm{Col}_2 - \mathrm{Col}_3 + \mathrm{Col}_4 = 0 ,$$

with coefficients $(1, -1, -1, 1)$ and a positive translate $(2, 0, 0, 2)$. Therefore any word w with two A's and two D's generates a 1-regular $L(w)$, even if w is not 0-regular.

(1) $L(\texttt{ADAD})$ is 1-regular, but $\texttt{ADAD}$ is not 0-regular.
(2) $L(\texttt{BCCBADDA})$ is 2-regular but $\texttt{BCCBADDA}$ is only 0-regular.

When we first understood the first assertion of Theorem 3.7 we hoped it would generate all the solutions to our particular Prouhet-Tarry-Escott problems. The third assertion dashed those hopes, so we started to search for other constructions. You can read about that in the next section. We close this one with some observations providing examples where $\mathcal{E}(L)$ is singular or invertible.

Notice that $\mathcal{E}$ has order three: $\mathcal{E}(\mathcal{E}(\mathcal{E}(L))) = L$ because

$$\mathcal{E}(\mathcal{E}(\mathcal{E}(L)))_{i,j} = x \Leftrightarrow \mathcal{E}(\mathcal{E}(L))_{j,x} = i \Leftrightarrow \mathcal{E}(L)_{x,i} = j \Leftrightarrow L_{i,j} = x .$$

This periodicity allows us to reduce the problem of finding Latin squares for which $\mathcal{E}(L)$ is singular or invertible to finding Latin squares with those properties.

Theorem 3.8. *For every positive integer n there exist invertible Latin squares of size n.*

Proof. Construct a Latin square M_n of size n by replacing k by $k+1$ in the usual addition table of the group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. After reversing the order of rows the corresponding matrix becomes a circulant matrix with first row $(n, 1, 2, \ldots, n-1)$, and

$$| \det M_n | = \frac{(n + 1)n^{n-1}}{2} \neq 0 \, ,$$

hence M_n is invertible. $\square$

For example, when $n = 6$ the Latin square M_6 is

(3.5)

	0	1	2	3	4	5
0	1	2	3	4	5	6
1	2	3	4	5	6	1
2	3	4	5	6	1	2
3	4	5	6	1	2	3
4	5	6	1	2	3	4
5	6	1	2	3	4	5

Theorem 3.9. *Let n be a composite positive integer. Then there exist singular Latin squares L of size n.*

Proof. Let a, b be integers such that $n = ab$ and $1 < a \leqslant b$. Consider the addition table M of the group $\mathbb{Z}_a \times \mathbb{Z}_b$. Enumerate the elements so that (i, j) is the $(j+1+bi)^{th}$. Then

$$\mathrm{Col}_1 - \mathrm{Col}_2 - \mathrm{Col}_{b+1} + \mathrm{Col}_{b+2} = 0 \, ,$$

hence M is not invertible. $\square$

The Latin square (3.4) corresponds to $a = b = 2$. When $a = 2$, $b = 3$ we obtain the singular normalized Latin square

(3.6)

	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$(0,0)$	1	2	3	4	5	6
$(0,1)$	2	3	1	5	6	4
$(0,2)$	3	1	2	6	4	5
$(1,0)$	4	5	6	1	2	3
$(1,1)$	5	6	4	2	3	1
$(1,2)$	6	4	5	3	1	2

where $\mathrm{Col}_1 + \mathrm{Col}_5 = \mathrm{Col}_2 + \mathrm{Col}_4$.

Remark 3.10. *Note that whether the addition table of a group is an invertible matrix or not depends on the order in which the elements are listed. Even though $\mathbb{Z}_6$ and $\mathbb{Z}_2 \times Z_3$ are isomorphic groups, the reordering of elements that maps (3.5) to (3.6) does not correspond to a group isomorphism.*

What happens when n is prime? There are no singular Latin squares of sizes 2 and 3 and a computer search indicates that all Latin squares of size 5 are invertible, too. However, for $n = 7$, the Latin square

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 6 & 5 & 4 & 3 & 1 \\ 3 & 6 & 7 & 2 & 1 & 4 & 5 \\ 4 & 5 & 2 & 1 & 6 & 7 & 3 \\ 5 & 1 & 4 & 7 & 3 & 2 & 6 \\ 6 & 4 & 1 & 3 & 7 & 5 & 2 \\ 7 & 3 & 5 & 6 & 2 & 1 & 4 \end{bmatrix}$$

is singular.

4. Changing the shapes of solutions

In this section we study regularity-preserving operations on words.

Theorem 4.1. *Swap. Let v, w, x, y and z be words on a b-letter alphabet such that v and w are $(r-1)$-regular and the concatenation $xvywz$ is r-regular. Suppose either*

- $|v| = |w|$, *or*
- y *is $(r-1)$-regular (possibly empty).*

Then $xwyvz$ is also r-regular.

Proof. Left to the reader. □

Theorem 4.2. *There are 1-regular words of length n on a two letter alphabet if and only if $n = 4k$. In that case every element of $\mathrm{PTE}(4k, 2, 1)$ can be obtained from the word*

$$w = \mathtt{A}^k \mathtt{B}^{2k} \mathtt{A}^k$$

by a sequence of swaps interchanging subwords $\mathtt{AB}$ and $\mathtt{BA}$.

Proof. Let v be a 1-regular word of length n on a two letter alphabet. Then n is even and since v is 1-regular, the block sums are equal, so

$$2(\text{block sum}) = \Sigma[n] = \frac{n(n+1)}{2} = \frac{n}{2} \times \text{odd} .$$

Then $n/2$ must also be even.

Conversely, it is clear that w is a 1-regular word of length $n = 4k$.

If the 1-regular word $v = \texttt{...BA...AB...}$ contains a subword $\texttt{BA}$ to the left of an $\texttt{AB}$ then Theorem 4.1 says $v' = \texttt{...AB...BA...}$ is also 1-regular and is strictly less that v in lexicographic order. We can repeat this procedure only a finite number of times, until we reach a 1-regular word z with no subwords $\texttt{BA}$ to the left of an $\texttt{AB}$. Then z is of the form $\texttt{A}^{p-1}\texttt{B}^q\texttt{AB}^{2k-q}\texttt{A}^{2k-p}$ for some $1 \leqslant p \leqslant 2k$ and $0 \leqslant q \leqslant 2k$. A straightforward computation shows that such a word is 1-regular if and only if $q = 2k(p-k)$, hence $q = 0, p = k$ or $q = 2k, p = k+1$. Both imply $z = w$. Reversing the sequence of swaps changes w into v. $\square$

Swapping rearranges a word without changing either length or regularity. Concatenation increases length, while preserving regularity:

Lemma 4.3. *If words v and w correspond to r-regular partitions on a b-letter alphabet then so does their concatenation vw.*

Proof. Let m be the length of v and n the length of w. Lemma 2.4 shows that shifting word w right by m gives an r-regular partition of the integers between $n+1$ and $n+m$. The blocks of the partition corresponding to vw are the unions of corresponding blocks of v and w. Since the component blocks from each of v and w have the same sums of powers up to r, so do their unions. $\square$

Splitting is the inverse of concatenation.

Definition 4.4. *(k-split) Let w be an r-regular word on a b-letter alphabet – that is, $w \in \text{PTE}(m, b, r)$. A k-split of w is a list of k-regular words $(w_1, w_2, \ldots, w_t)$ such that $w = w_1 w_2 \cdots w_t$.*

The words w_i need not have the same length. Lemma 4.3 implies that if w can be k-split, then it is k-regular.

Example 4.5. *We can k-split the familiar 2-regular* $\texttt{ABBABAAB}$ *several ways – the blanks illustrate the subword boundaries:*

$$\texttt{ABBABAAB} = \texttt{ABBA BAAB} = \texttt{ABBA BA AB} = \texttt{AB BA BA AB}.$$

Theorem 4.1 implies that reordering the pieces of an $(r-1)$-splitting of a partition of regularity r does not alter the regularity below r.

Definition 4.6. *Let $(w_1, w_2, \ldots, w_t)$ be a list of words of the same length on the same alphabet. The* shuffle

$$w_1 \wedge w_2 \wedge \ldots \wedge w_t$$

of the list is the word w built by concatenating the words built by concating the t first, second, $\ldots$ letters of the w_i.

Example 4.7.

$$AB \wedge BC \wedge CA = ABC \ BCA$$
$$ABBA \wedge BAAB = ABBABAAB$$
$$ABBA \wedge ABBA = AABBBBAA$$

Theorem 4.8. *(Shuffling) The shuffle of r-regular words is r-regular.*

Proof. Each component appears in the shuffle as an affine shift. ☐

Swapping, concatenation and shuffling are all methods of generating new regular words from old. We have introduced these operations in hopes that they will help find all the regular words from some known ones, by analogy with Theorem 4.2. There may be interesting questions to ask and answer about the algebra of these operations – the ways in which they associate, commute and distribute.

5. EXISTENCE

Theorem 5.1. *On a two-letter alphabet, there are 2-regular words of length $n \iff n = 4k$, with $k \geqslant 2$.*

Proof. Suppose there are 2-regular words of length n. Theorem 4.2 implies that $n = 4k$, since any 2-regular word is 1-regular. There are no 2-regular words of length 4, hence $k \geqslant 2$.

Conversely, suppose $n = 4k$ with $k \geqslant 2$. Then k can be written as a sum of $2s$ and $3s$, hence some concatenation of copies of the 2-regular words ABBABAAB and ABABBBAAABAB generate a 2-regular $4k$-letter word. ☐

The 12-letter word ABABBBAAABAB is a mystery. A computation similar to the one following Example 3.4 shows it is 2-regular:

$$
\begin{aligned}
X &= 1^2 + 3^2 + 7^2 + 8^2 + 9^2 + 11^2 \\
&= 1^2 + 3^2 + (1+6)^2 + (2+6)^2 + (3+6)^2 + (2+9)^2 \\
&= 1^2 + 3^2 + 1^2 + 2(1 \times 6) + 6^2 + 2^2 + 2(2 \times 6) + 6^2 \\
&= 3^2 + 2(3 \times 6) + 6^2 + 2^2 + 2(2 \times 9) + 9^2 \\
&= 2(1^2 + 2^2 + 3^2) + 6(2 + 4 + 6 + 6) + (3 \times 6^2 + 9^2)
\end{aligned}
$$

while

$$Y = 2(1^2 + 2^2 + 3^2) + 6(1 + 2 + 3 + 3 + 9) + (3 \times 3^2 + 2 \times 9^2).$$

The word is 2-regular because these expressions are equal – term by term. Why does that happen?

Theorem 5.2. *Let $r \geqslant 2$ and $n = k \cdot 2^r$, with $k \geqslant 2$. Then there exist r-regular words of length n over a two-letter alphabet.*

Proof. Induction on r. The base case $r = 2$ is in Theorem 5.1. The induction step follows from Theorem 3.7.   □

A computer search shows that $PTE(2, 2, 0)$, $PTE(4, 2, 1)$, $PTE(8, 2, 2)$, $PTE(16, 2, 3)$ each contain just one word, the initial segment of the Thue-Morse sequence of the corresponding length. Moreover, those are the minimal lengths of words with the respective regularity.

Conjecture 5.3. *Suppose $r \geqslant 2$. On a two-letter alphabet, there are r-regular words of length $n \iff n = k \cdot 2^r$, with $k \geqslant 2$. Moreover, $PTE(2^{r+1}, 2, r)$ contains just one word, the initial segment of the Thue-Morse sequence of length 2^{r+1}.*

There are similar results for three-letter alphabets.

Theorem 5.4. *On a three-letter alphabet:*
  (1) *There are 1-regular words of length $n \iff n = 3k$, with $k \geqslant 2$.*
  (2) *There are 2-regular words of length $n \iff n = 9k$, with $k \geqslant 2$.*

Proof. Similar to the proof of the first part of Theorem 4.2.   □

A computer search shows that $PTE(6, 3, 1)$ has one word (`ABCCBA`), $PTE(18, 3, 2)$ has nine words, and $PTE(36, 3, 3)$ has 152. Those are the minimum lengths of words of regularity 1, 2, and 3 respectively. These numbers show that:
  (1) There are 2-regular words of length 18 that do not come from a Latin square construction starting with a 1-regular word of length 6.
  (2) None of the 3-regular words of length 36 comes from a Latin square construction starting with a word of length 12, since the Latin squares of order 3 are invertible and there are no 2-regular words of length 12.

Theorem 5.5. *Let $r \geqslant 3$ and $n = 2 \cdot k \cdot 3^{r-1}$ with $k \geqslant 2$. Then there exists r-regular words of length n over a three-letter alphabet.*

Proof. Induction on r. For $r = 3$ there are 3-regular words of $36 = 18 \cdot 2$ and $54 = 18 \cdot 3$ letters, hence, by concatenation, of any length of the form $18k$ with $k \geqslant 2$. The induction step follows from Theorem 3.7.   □

6. RESOURCE ALLOCATION

How does all this help answer the question of three or more cups of coffee? We model the concentration of coffee in a cylindrical cafetière

as a function $f \colon [0,1] \to \mathbb{R}$. (In reality f will increase with depth, but we won't need that.) To fill b cups of coffee with m pours of equal size we want to choose a partition $\{B_1, \ldots, B_b\}$ of the set of subintervals

$$(6.1) \qquad \left\{ \left[0, \frac{1}{m}\right], \left[\frac{1}{m}, \frac{2}{m}\right], \ldots, \left[\frac{m-1}{m}, 1\right] \right\}$$

such that the integrals

$$(6.2) \qquad c_j = \int_{B_j} f(x)dx = \sum_{I \in B_j} \int_I f(x)dx$$

are as nearly equal as possible.

We will identify the intervals in (6.1) by m times their right endpoints, so the partitions of that set of intervals are just the partitions of $\{1, 2, \ldots, m\}$ we have been studying.

Theorem 6.1. *If $B \in \mathrm{PTE}(m, b, r)$ then the integrals in Equation (6.2) are independent of j when f is a polynomial of degree at most r. Therefore B is a perfect pouring.*

Proof. Consider first a monomial $f(x) = x^n$ for $n \leqslant r$. Using the change of variable $y = mx$ we have

$$c_j = \frac{1}{m^{n+1}} \sum_{i \in B_j} \int_{i-1}^{i} y^n dy = \frac{1}{(n+1)m^{n+1}} \sum_{i \in B_j} \left(i^{n+1} - (i-1)^{n+1} \right).$$

But $i^{n+1} - (i-1)^{n+1}$ is a polynomial of degree n in i and since B is r-regular and $n \leqslant r$, the last sum is independent of j. Having proved the theorem for monomials its truth follows easily for polynomials. $\square$

This argument may seem circular. It's not: the theorem asserts the equality of integrals of sums of powers; the last part of the last paragraph uses regularity to prove the equality of sums of sums of powers.

When f is not a polynomial we can use the first few terms of its Taylor expansion to find pretty good pourings.

Theorem 6.2. *Let $f \colon [0,1] \to \mathbb{R}$ be an $r+1$-times differentiable function and suppose $|f^{(r+1)}(x)| \leqslant M$ for all $0 \leqslant x \leqslant 1$. If $B \in \mathrm{PTE}(m, b, r)$ then*

$$(6.3) \qquad |c_i - c_j| \leqslant \frac{M}{2^r b(r+1)!} \ .$$

Proof. The Lagrange formula for the remainder of the Taylor expansion of f about $1/2$ says that

$$f(x) = \text{a polynomial of degree } r + R(x)$$

where the error term satisfies

$$|R(x)| = \left| \frac{f^{(r+1)}(\xi_x)}{(r+1)!} \left(x - \frac{1}{2} \right)^{r+1} \right| \leqslant \frac{M}{2^{r+1}(r+1)!}$$

for some ξ_x between 0 and 1. Then

$$(6.4) \quad c_i - c_j = \int_{B_i} f(x)dx - \int_{B_j} f(x)dx = \int_{B_i} R(x)dx - \int_{B_j} R(x)dx$$

because the polynomial parts of the expansion of f contribute the same amount to the difference. Each of the two terms in (6.4) satisfies the inequality

$$\left| \int_{B_i} R(x)dx \right| \leqslant \frac{M}{2^{r+1}b(r+1)!}$$

since B_i is the union of m/b intervals each of length $1/m$. Then their difference satisfies (6.3). □

Example 6.3. *Suppose $f(x) = e^{-ax}$, with $a > 0$. Then*

$$\left| f^{(r)}(x) \right| = \left| (-a)^r e^{-ax} \right| \leqslant a^r .$$

Then the right side of (6.3) approaches 0 as $r \to \infty$, so we have a strategy for pouring as equitably as we wish by choosing a PTE solution with r large enough.

The inequality in (6.3) provides a quantitative estimate of the error of a particular pouring. Here is a more general qualitative assertion:

Theorem 6.4. *Suppose $f\colon [0,1] \to \mathbb{R}$ is analytic. Then we can get a pouring as close to equitable as we want by choosing a partition in* PTE(m,b,r) *for r large enough.*

Proof. The difference in remainders in Equation (6.4) can be made arbitrarily small since f is the uniform limit of the partial sums of its power series. □

In [LS12] the authors address resource allocations for two players and remark that "It would be interesting to quantify the intuition that the Thue-Morse order tends to produce a fair outcome." Theorem 6.4 and Conjecture 5.3 show that allocations tend to be more equitable as regularity increases, and that the Thue-Morse sequence produces the highest regularity for words of fixed lengths that are powers of 2.

In [Ric01] Richman showed that the Thue-Morse sequence provides the most equitable pourings into two cups for a variety of density functions f. Our analysis here does not extend his; all we show is that regular partitions yield good pourings.

Should you ever actually use a regular partition for a pouring you can take advantage of double letters in the word to save a few switches: `ABBABAAB` requires just 5, not 7. But don't get your hopes up. The Thue-Morse sequence never contains `xxx`. That's probably true for our generalizations, too. Nor are you likely to find `xxyyzz`.[1]

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[1] "bookkeeper" is essentially the only English word we know that does.